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The Jacobson radical of rings with nilpotent homogeneous elements

Agata Smoktunowicz

ABSTRACT

A result of Bergman says that the Jacobson radical of a graded algebra is homogeneous. It is shown that while graded Jacobson radical algebras have homogeneous elements nilpotent, it is not the case that graded algebras all of whose homogeneous elements are nilpotent are Jacobson radical. To contrast this, the following result of the author is slightly extended. Let R be a graded algebra generated in the degree one. If for every n , the $n \times n$ matrix algebra over R has all homogeneous elements nilpotent, then R is Jacobson radical.

1. Introduction

The Jacobson radical is an important tool for studying the structure of non-commutative algebras. It is related to the Hopkins–Levitzki theorem, Nakayama’s lemma, Jacobson’s density theorem, to name just a few [18]. The Jacobson radical of a ring R , denoted by $J(R)$, is the intersection of all maximal left ideals in R . On the other hand $J(R)$ is the largest ideal in R consisting of quasi-invertible elements. A left quasi-inverse of an element $r \in R$ is an element $s \in R$, such that $r + s - sr = 0$. A ring equal to its Jacobson radical is called a Jacobson radical ring. For example, every nil ring (a ring all of whose elements are nilpotent) is Jacobson radical. Amitsur proved that finitely generated Jacobson radical algebras over uncountable fields are nil [18]. Also Jacobson radicals of algebraic algebras are nil.

A sum of two Jacobson radical one-sided ideals in a ring is Jacobson radical [9]. Matrix rings over Jacobson radical rings are Jacobson radical [18]. It is not known if similar results hold for nil rings. This question is related to the famous Koethe conjecture (1930) which states that a ring without non-zero nil ideals has no non-zero one-sided nil ideals (see [10, 14, 27]). Koethe’s conjecture is still open, even for graded algebras. Krempa showed that the Koethe conjecture is equivalent to the assertion that for every nil ring R the 2×2 matrix ring over R is nil [16]. In the same paper he proved that the conjecture is equivalent to the statement that polynomial rings over nil rings are Jacobson radical. Notice that polynomial rings over nil rings need not be nil [29]. It is known that the Jacobson radical of some important classes of algebras is nil [3, 6, 8, 13, 25]. For general information about nil rings we refer the reader to [34]. In 1972 Krempa proved that if N is a ring and all matrices with coefficients in N are nilpotent then the polynomial ring over N is Jacobson radical [16]. Note that another proof was later found by Amitsur (see [18, p. 171]). This result was generalized by the author in 2004 and quoted in [2] with a sketch of a proof.

THEOREM 1.1 (Smoktunowicz [2]). *Let $R = \bigoplus_{i=1}^{\infty} R_i$ be a graded algebra generated in degree one. Then the following statements are equivalent.*

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- (i) For all natural numbers n, m , all $n \times n$ matrices with entries from R_m are nilpotent.
- (ii) For every natural number n , all $n \times n$ matrices with entries from R_1 are nilpotent.
- (iii) The graded algebra R is Jacobson radical.

It is not known if properties (i) and (iii) are equivalent without the assumption that the ring R is generated in degree one. Theorem 1.1 can be extended to ungraded rings as follows.

THEOREM 1.2. *Let R be a ring, S be a subset of R , and let $P = S + S^2 + \dots$ be a subring of R generated by S . Suppose that all $n \times n$ matrices with coefficients from S are nilpotent, for $n = 1, 2, \dots$. Then the following are true.*

- (i) For all natural numbers n, m , all $n \times n$ matrices with entries from S^m are nilpotent.
- (ii) The ring P is Jacobson radical.

Graded algebras with nilpotent homogeneous elements are related to algebraic division algebras and to some groups. Lie algebras constructed from Grigorchuk groups are nil and have Gelfand–Kirillov dimension strictly between one and two (see [23] for fields of characteristic two and [28] for fields of other characteristic). For general information about the Gelfand–Kirillov dimension we refer the reader to [15]. A recurrent transitive algebra constructed from the Grigorchuk group of intermediate growth is graded and Jacobson radical and hence has all homogeneous elements nilpotent, provided that the base field is an algebraic extension of a finite field of characteristic two. If the base field is not an algebraic extension of F_2 , such algebras are semiprimitive, and it is an open question whether they are primitive [2].

On the other hand, graded algebras associated to algebraic division algebras have all homogeneous elements nilpotent, and if the base field is uncountable, they are nil [26].

Montgomery and Small proved that Noetherian graded algebras with nilpotent homogeneous elements are nilpotent [22]. In the case when an affine algebra R is ungraded the Jacobson radical of R may have no nilpotent elements, even if R has a finite Gelfand–Kirillov dimension [4]. A result of Bergman says that the Jacobson radical of a graded algebra is homogeneous [7]. The main result of this paper is the following.

THEOREM 1.3. *Over every field K , there is a graded algebra $R = \bigoplus_{i=1}^{\infty} R_i$, generated by two elements of degree one, which has all homogeneous elements nilpotent and is not Jacobson radical.*

This answers in the affirmative a question of Bartholdi (Private Communication, Lausanne, November 2002), and in the case of uncountable fields of Small and Zelmanov (Private Communication, Oberwolfach, May 2006).

All the mentioned rings and algebras are associative and non-commutative. In Sections 1–5 we will use the following notations.

- (1) Let A be the algebra of polynomials in non-commuting indeterminates x, y over a field K .
 - (2) Let M denote the set of all monomials in x, y , and for each integer $n \geq 1$, let $M(n)$ denote the set of monomials of degree n . Thus $M(0) = \{1\}$, and for $n \geq 1$ the elements in $M(n)$ are of the form $x_1 \dots x_n$, where $x_i \in \{x, y\}$.
 - (3) The K subspace of A spanned by $M(n)$ will be denoted by $H(n)$, and elements of $H(n)$ will be called homogeneous polynomials of degree n .
 - (4) The degree of element $r \in R$ is the smallest number $d = d(r)$ such that $r \in \sum_{i=0}^d H(i)$.
- In Sections 1–5, Theorem 1.3 is proved. To do this we introduce linear mappings F_n . In Sections 6–7, we prove Theorem 1.2. In Section 8, we discuss some open questions.

2. Linear mappings

In this section we introduce linear mappings F_n . In the next section we will construct an ideal I such that R/I has nilpotent homogeneous elements and I belongs to the union of the kernels of the mappings F_n . Let $c_0 = 25$ and define inductively, for $n > 0$,

$$c_n = n(4c_{n-1} + 1).$$

We shall define inductively for $n = 0, 1, \dots, K$ -linear mappings

$$\begin{aligned} E_{n+1} &: H(c_n(n+1)) \longrightarrow H(c_n(n+1)), \\ G_{n+1} &: H((2c_n+1)(n+1)) \longrightarrow H((2c_n+1)(n+1)), \\ F_{n+1} &: H(c_{n+1}) \longrightarrow H(c_{n+1}). \end{aligned}$$

We first define them for monomials and then extend them by linearity.

Let $E_0 = \text{id}$, $G_0 = \text{id}$, $F_0 = \text{id}$. Suppose that $n \geq 0$ and we have already defined E_j, G_j, F_j for $j \leq n$.

Let $v \in M(c_n(n+1))$. We can write $v = \prod_{i=1}^{n+1} v_i$ for some $v_i \in M(c_n)$. Define

$$E_{n+1}(v) = E_{n+1} \left(\prod_{i=1}^{n+1} v_i \right) = \prod_{i=1}^{n+1} F_n(v_i).$$

Let $u \in M((2c_n+1)(n+1))$. We can write $u = \prod_{i=1}^{n+1} s_i p_i x_i$ for some $s_i, p_i \in M(c_n)$, $x_i \in \{x, y\}$. Define

$$G_{n+1}(u) = G_{n+1} \left(\prod_{i=1}^{n+1} s_i p_i x_i \right) = \prod_{i=1}^{n+1} [F_n(s_i)F_n(p_i) - F_n(p_i)F_n(s_i)]x_i.$$

Given $w \in M(c_{n+1})$, write

$$w = w_1 w_2 w_3,$$

where $w_1, w_3 \in H((n+1)c_n)$, $w_2 \in H((2c_n+1)(n+1))$

$$F_{n+1}(w) = F_{n+1}(w_1 w_2 w_3) = E_{n+1}(w_1) G_{n+1}(w_2) E_{n+1}(w_3).$$

Moreover, given a natural number m , denote the set S_m as follows:

$$S_m = \{i : i = kc_{m-1} \text{ or } i = mc_{m-1} + k(2c_{m-1} + 1)\}$$

or

$$\begin{aligned} i &= c_{m-1} + mc_{m-1} + k(2c_{m-1} + 1) \\ i &= kc_{m-1} + (3c_{m-1} + 1)m \text{ for some } 0 \leq k \leq m. \end{aligned}$$

Given a subset S of A , denote $ASA = \{\sum_i a_i s_i b_i : a_i, b_i \in A, s_i \in S\}$.

LEMMA 2.1. Let $m > n \geq 0$ be integers. Let $w \in H(k)dH(j) \subseteq H(c_m)$, where $k \in S_m$ and $d \in H(c_{m-1})$. Then the following assertions are true.

- (1) $F_m(w) \in AF_{m-1}(d)A$.
- (2) If $j = 0$ then $F_m(w) \in AF_{m-1}(d)$ and if $k = 0$ then $F_m(w) \in F_{m-1}(d)A$.
- (3) Let $e \in H(c_{n+1})$. If $d \in H(c_m - c_{n+1})e$ or $d \in eH(c_m - c_{n+1})$ then $F_m(w) \in AF_{n+1}(e)A$.

Proof. (1) Because the mappings F_m are linear, it suffices to consider the case when $w \in M(i)dM(j)$. Write $w = w_1 w_2 w_3$, where $w_1 = \prod_{i=1}^m v_i$, $w_3 = \prod_{i=1}^m \bar{v}_i$, $w_2 = \prod_{i=1}^m s_i p_i x_i$, and where all $v_i, \bar{v}_i, s_i, p_i \in H(c_{m-1})$, $x_i \in H(1)$. Observe that there is an $i \leq m$ such that either $v_i = d$, $\bar{v}_i = d$, $s_i = d$, or $p_i = d$, because $k \in S_m$. By the definition of F_m , we get that $F_m(w) \in AF_{m-1}(d)A$, as required.

(2) Observe that, with the notation as in (1), if $j = 0$ then $\bar{v}_m = d$. Therefore $F_m(w) \in AF_{m-1}(d)$, by the definition of the mapping F_m .

If $k = 0$ then $v_1 = d$. Therefore $F_m(w) \in F_{m-1}(d)A$, by the definition of the mapping F_m .

(3) If $n + 1 = m$ then the result is true by (1). If $n + 1 < m$ then we get, by applying (2) several times, that either $F_{m-1}(d) \in AF_{n+1}(e)$ or $F_{m-1}(d) \in F_{n+1}(e)A$. Now $F_m(w) \subseteq AF_{n+1}(e)A$. \square

3. The ideal of defining relations

Let I be the ideal generated by all elements from the set

$$\{r_i^{c_i+5} : r_i \in H(i), i = 1, 2, \dots\}.$$

Note that I contains powers of all homogeneous elements in R . In this section we prove that if $r \in R$ and $F_n(r) \neq 0$, for some n , then $r \notin I$.

LEMMA 3.1. *Let n be a natural number and let $a \in H(n)$. Let e, f be monomials of degrees not exceeding n . If j is a natural number such that $ea^j f \in H(c_{n+1})$ then $F_{n+1}(ea^j f) = 0$.*

Proof. Observe first that n divides c_n for any $n > 0$ and let $k = c_n/n$. Notice that $a^k \in H(c_n)$. Recall that $c_{n+1} = (4c_n + 1)(n + 1) > c_n(3n + 3)$. Observe that $a^j \in H(t)a^{2k}A$, where $t = c_n(n + 1 + 2 \deg e)$. Therefore $ea^j f \in H(\alpha)a^{2k}H(l)$, where $\alpha = t + \deg e = c_n(n + 1) + \deg e(2c_n + 1)$, for some l . By the definition of the mapping F_n , we get

$$F_{n+1}(H(\alpha)a^{2k}H(l)) \subseteq H(\alpha)[F_n(a^k)F_n(a^k) - F_n(a^k)F_n(a^k)]H(l).$$

Hence $F_{n+1}(ea^j f) = 0$, as required. \square

LEMMA 3.2. *Let m be a natural number. If $w \in I \cap H(c_m)$ then $F_m(w) = 0$.*

Proof. We proceed by induction on m . If $m = 0$ then $w = 0$ and the result holds. Assume that the lemma is true for all numbers smaller than m . Let n be a natural number, let $a \in H(n)$, and let $w = pa^{c_{n+5}}q \in H(c_m)$ for some monomials p, q . We will show that $F_m(w) = 0$. Let S_m be as in Section 2. Then there are $p', p'' \in M$ such that $p = p'p''$, $\deg p' \in S_m$ and $\deg p'' \leq c_{m-1}$. We have two possibilities.

Case 1. Assume that $\deg(p''a^{c_{n+5}}) \leq c_{m-1}$. There are $q', q'' \in M$ such that $p''a^{c_{n+5}}q' \in H(c_{m-1})$ and $q = q'q''$. Observe that $p = p'p''a^{c_{n+5}}q'q'' \in H(\deg p')(p''a^{c_{n+5}}q')H(\deg q'')$. By Lemma 2.1 applied with $k = \deg p'$, $w = p''a^{c_{n+5}}q'$ we get

$$F_m(w) = F_m(p'(p''a^{c_{n+5}}q')q'') \in AF_{m-1}(p''a^{c_{n+5}}q')A.$$

By the inductive assumption $F_{m-1}(p''a^{c_{n+5}}q') = 0$, and so $F_m(w) = 0$.

Case 2. Assume that $\deg(p''a^{c_{n+5}}) > c_{m-1}$. There is a $k \in S_m$ such that $0 < k - \deg p' \leq c_{m-1} + 1$. Note that $\deg pa^{c_{n+5}} \geq k$. If $\deg pa^{c_{n+3}} \leq k$ then $pa^{c_{n+5}}A \subseteq H(\deg p')dA$, where $d \in p''a^jH(i) \subseteq H(c_{m-1})$ for some $i \leq n$ and $j \geq c_{n+3}$.

Recall that $w = pa^{c_{n+5}}q \in H(c_m)$. It follows that $n + 5 \leq m$. If $\deg pa^{c_{n+3}} \geq k$ then $pa^{c_{n+5}}A \subseteq H(k)d'A$, where $d' \in H(i)a^jA \subseteq H(c_{m-1})$ for some $i \leq n$ and $j \geq c_{n+3}$ (since $nc_{n+3} + n \leq c_{m-1}$ and $2c_{n+3} + 2 < c_{n+5}$). By Lemma 2.1(1) either $F_m(w) \in AF_{m-1}(d)A$ or $F_m(w) \in AF_{m-1}(d')A$.

Observe now that there is an $e \in \sum_{0 \leq i, j \leq n} H(i)a^{p(i,j)}H(j) \subseteq H(c_{n+1})$, for some numbers $p_{i,j}$, such that either $d \in H(c_{m-1} - c_{n+1})e$ or $d' \in eH(c_{m-1} - c_{n+1})$. By Lemma 2.1(3) either $F_{n-1}(d) \in AF_{n+1}(e)A$ or $F_{n-1}(d') \in AF_{n+1}(e)A$. By Lemma 3.1, $F_{n+1}(e) = 0$, and so $F_m(w) = 0$, as required. \square

4. Quasi-regular elements

In R , define $a * b = a + b - ab$. The operation $*$ is associative, and 0 is the identity element. An element $a \in R$ is called left quasi-regular if a has left inverse in the monoid $(R, *)$. A ring all of whose elements are left quasi-regular is Jacobson radical.

Let $w(i)$ be the i th homogeneous component of $(1 - x - y^2)^{-1}$. Observe that $w(0) = 1$, $w(1) = x$, and $w(2) = y^2 + x^2$.

LEMMA 4.1. *Let $i \geq 2$; then*

$$\begin{aligned} w(i) &= xw(i-1) + y^2w(i-2), \\ w(i) &= w(i-1)x + w(i-2)y^2. \end{aligned}$$

Proof. This follows from the fact that $w(i)$ is the i th homogeneous component of $(1 - x - y^2)^{-1}$. \square

LEMMA 4.2. *Let $k < i$ be natural numbers. Then*

$$w(i) = w(k)w(i-k) + w(k-1)y^2w(i-k-1).$$

Proof. We prove the lemma by induction on k . If $k = 1$ the result is true by the definition of elements $w(n)$. Suppose the result is true for some $k < i - 1$. We will show that $w(i) = w(k+1)w(i-k-1) + w(k)y^2w(i-k-2)$. By the definition, $w(i-k) = xw(i-k-1) + y^2w(i-k-2)$. By the assumption, $w(i) = w(k)w(i-k) + w(k-1)y^2w(i-k-1)$. Therefore

$$w(i) = w(k)xw(i-k-1) + w(k)y^2w(i-k-2) + w(k-1)y^2w(i-k-1).$$

By Lemma 4.1, we have $w(k)x + w(k-1)y^2 = w(k+1)$, and so $w(i) = w(k+1)w(i-k-1) + w(k)y^2w(i-k-2)$, as required. \square

LEMMA 4.3. *Let $k < i$ be natural numbers. Then*

$$w(i) \in w(k)xw(i-k-1) + H(k)yH(i-k-1).$$

Proof. It follows from the previous lemma, because $w(i-k) = xw(i-k-1) + y^2w(i-k-2)$. \square

LEMMA 4.4. *Let $p, s \geq 1$. Then $w((p+1)s) = v + \prod_{i=1}^s w(p)x$, for some $v \in \sum_{j=0}^{s-1} H(j(p+1) + p)yA$.*

Proof. The proof is by induction on s . If $s = 1$ then $w(p+1) \in w(p)x + H(p)y$, by Lemma 4.1. Suppose the result is true for some s , we will show it is true for $s+1$. By

Lemma 4.3, $w((p+1)(s+1)) \in w(p)xw((p+1)s) + H(p)yA$. By the inductive assumption, $w((p+1)s) \in \prod_{i=1}^s w(p)x + v$. Therefore $w((p+1)(s+1)) \in \prod_{i=1}^s w(p)x + H(p)yA + H(p+1)v$, as required. \square

LEMMA 4.5. *Let T be a homogeneous ideal in A . Suppose that there is an $a \in A$ such that $(1-x-y^2)(1+a)-1 \in T$. Then $w(n) \in T$ for almost all n .*

Proof. Let $a = a_1 + a_2 + \dots + a_s$, with $a_i \in H(i)$. We can write $a = a_1 + a_2 + \dots + a_s + a_{s+1} + a_{s+2}$, with $a_i \in H(i)$ and $a_{s+1} = a_{s+2} = 0$. Observe that $a_1 - x \in T$, $a_2 - x^2 - y^2 \in T$, and $a_i - xa_{i-1} - y^2a_{i-2} \in T$, since T is homogeneous and $(1-x-y^2)(1+a)-1 \in T$. Therefore $a_i - w(i) \in T$ for all i . It follows that

$$\begin{aligned} w(s+1) &= xw(s) + y^2w(s-1) \in xa_s + y^2a_{s-1} + T = a_{s+1} + T = T, \\ w(s+2) &= xw(s+1) + y^2w(s) \in xa_{s+1} + y^2a_s + T = a_{s+2} + T = T. \end{aligned}$$

Hence $w(n) \in T$ for all $n > s$, because $w(n) = xw(n-1) + y^2w(n-2)$ by the definition of $w(n)$ (by induction on n). \square

5. Theorem 1.3

In Sections 2–4 we introduced the elements $w(n)$, the mappings F_n , and the ideal I . In Section 3 it was shown that if $F_n(r) \neq 0$, for some n , then $r \notin I$. The aim of this section is to show that $F_n(w_n) \neq 0$, and so $w_n \notin I$ for all n . This and Lemma 4.5 are then used to prove Theorem 1.3. Given a number $n \geq 0$, denote

$$\begin{aligned} \alpha(n) &= w(c_n), & \beta(n) &= w(c_n - 1)y, & \gamma(n) &= yw(c_n - 1), & \xi(n) &= yw(c_n - 2)y, \\ a_n &= w((n+1)c_n), & b_n &= w((n+1)c_n - 1)y, \\ e_n &= yw((n+1)c_n - 1), & d_n &= yw((n+1)c_n - 2)y. \end{aligned}$$

LEMMA 5.1. *Let $n \geq 0$. Assume that the elements $F_n(\alpha(n)), F_n(\beta(n)), F_n(\gamma(n)), F_n(\xi(n))$ are linearly independent over K . Then the elements $E_{n+1}(a_n), E_{n+1}(b_n), E_{n+1}(e(n)), E_{n+1}(d(n))$ are linearly independent over K .*

Proof. Observe first that all products of the form $t_1t_2\dots t_n$ with all $t_i \in \{F_n(\alpha(n)), F_n(\beta(n)), F_n(\gamma(n)), F_n(\xi(n))\}$ are linearly independent over K , because the elements $F_n(\alpha(n)), F_n(\beta(n)), F_n(\gamma(n)), F_n(\xi(n))$ are linearly independent over K and are of the same degree. Note that the elements a_n, b_n, e_n, d_n are linearly independent over K . Applying Lemma 4.2 several times shows that a_n, b_n, e_n, d_n are products of $\alpha(n), \beta(n), \gamma(n), \xi(n)$. Hence $E_{n+1}(a_n), E_{n+1}(b_n), E_{n+1}(e_n), E_{n+1}(d_n)$ are, respectively, products of $F_n(\alpha(n)), F_n(\beta(n)), F_n(\gamma(n)), F_n(\xi(n))$, and hence are linearly independent over K . \square

LEMMA 5.2. *Let $n \geq 0$. Assume that the elements $F_n(\alpha(n)), F_n(\beta(n)), F_n(\gamma(n)), F_n(\xi(n))$ are linearly independent over K . Then $G_{n+1}(a_n) \neq 0$.*

Proof. Observe first that by Lemma 4.4, applied for $p = 2c_n$ and $s = n+1$, $w((2c_n+1)(n+1)) = \prod_{i=1}^{n+1} w(2c_n)x + v$, where $v \in \sum_{j=0}^n H(j(2c_n+1) + 2c_n)yA$. By Lemma 4.2, $w(2c_n) = w(c_n)w(c_n) + w(c_n-1)yyw(c_n-1)$. By the definition of the mapping G_{n+1} we get

$G_{n+1}(w((2c_n + 1)(n + 1))) = f + g$, where

$$f = \prod_{i=1}^{n+1} [F_n(w(c_n - 1)y)F_n(yw(c_n - 1)) - F_n(yw(c_n - 1))F_n(w(c_n - 1)y)]x$$

and $g = G_{n+1}(v)$. Note that $G_{n+1}(v) \in \sum_{j=0}^n H(j(2c_n + 1) + 2c_n)yA$.

By the assumptions, $F_n(yw(c_n - 1)) = F_n(\gamma(n))$ and $F_{n-1}(w(c_{n-1})y) = F_n(\beta(n))$ are linearly independent over K , and so $f \neq 0$. Because the elements x, y are linearly independent over K , it follows that $G_{n+1}(w((2c_n + 1)(n + 1))) = f + g \neq 0$ (by comparing the elements on places of degrees $j(2c_n + 1) + 2c_n$, for $j = 1, 2, \dots, n$). \square

LEMMA 5.3. *For every natural number n , the elements $F_n(\alpha(n))$, $F_n(\beta(n))$, $F_n(\gamma(n))$, $F_n(\xi(n))$ are linearly independent over K .*

Proof. We prove the lemma by induction on n . If $n = 0$, then $R_n = \text{id}$ and the result follows. Suppose that the elements $F_n(\alpha(n))$, $F_n(\beta(n))$, $F_n(\gamma(n))$, $F_n(\xi(n))$ are linearly independent over K . We will show that $F_{n+1}(\alpha(n + 1))$, $F_{n+1}(\beta(n + 1))$, $F_{n+1}(\gamma(n + 1))$, $F_{n+1}(\xi(n + 1))$ are linearly independent over K .

By Lemma 4.2, we get $\alpha(n + 1) = a_nr_1 + b_nr_2$, $\beta(n + 1) = a_nr_3 + b_nr_4$, $\gamma(n + 1) = e_nr_5 + d_nr_6$, $\xi(n + 1) = e_nr_7 + d_nr_8$, for some $r_1, \dots, r_8 \in H(c_{n+1} - c_n(n + 1))$. By the definition of the mappings F_{n+1} we get that

$$\begin{aligned} F_{n+1}(\alpha(n + 1)) &= E_{n+1}(a_n)s_1 + E_{n+1}(b_n)s_2, \\ F_{n+1}(\beta(n + 1)) &= E_{n+1}(a_n)s_3 + E_{n+1}(b_n)s_4, \\ F_{n+1}(\gamma(n + 1)) &= E_{n+1}(e_n)s_5 + E_{n+1}(d_n)s_6, \\ F_{n+1}(\xi(n + 1)) &= E_{n+1}(e_n)s_7 + E_{n+1}(d_n)s_8, \end{aligned}$$

for some $s_1, \dots, s_8 \in H(c_{n+1} - c_n(n + 1))$.

Suppose that the elements $F_{n+1}(\alpha(n + 1))$, $F_{n+1}(\beta(n + 1))$, $F_{n+1}(\gamma(n + 1))$, $F_{n+1}(\xi(n + 1))$ are linearly dependent over K . Then

$$i_1F_{n+1}(\alpha(n + 1)) + i_2F_{n+1}(\beta(n + 1)) + i_3F_{n+1}(\gamma(n + 1)) + i_4F_{n+1}(\xi(n + 1)) = 0$$

for some $i_1, i_2, i_3, i_4 \in K$, not all equal to zero.

Notice that

$$\begin{aligned} F_{n+1}(\alpha(n + 1)), F_{n+1}(\beta(n + 1)) &\in E_{n+1}(a_n)A + E_{n+1}(b_n)A, \\ F_{n+1}(\gamma(n + 1)), F_{n+1}(\xi(n + 1)) &\in E_{n+1}(e_n)A + E_{n+1}(d_n)A. \end{aligned}$$

By the inductive assumption, $F_n(\alpha(n))$, $F_n(\beta(n))$, $F_n(\gamma(n))$, $F_n(\xi(n))$ are linearly independent over K . By Lemma 5.1, the elements $E_{n+1}(a_n)$, $E_{n+1}(b_n)$, $E_{n+1}(c_n)$, $E_{n+1}(d_n)$ are linearly independent over K . It follows that $i_1\alpha(n + 1) + i_2\beta(n + 1) = 0$ and $i_3\gamma(n + 1) + i_4\xi(n + 1) = 0$. Similarly, observe that

$$\begin{aligned} F_{n+1}(\alpha(n + 1)), F_{n+1}(\gamma(n + 1)) &\in AE_{n+1}(a_n) + AE_{n+1}(e_n), \\ E_{n+1}(\beta(n + 1)), E_{n+1}(\xi(n + 1)) &\in AE_{n+1}(b_n) + AE_{n+1}(d_n). \end{aligned}$$

Arguing as before, $i_1F_{n+1}(\alpha(n + 1)) + i_2F_{n+1}(\beta(n + 1)) = 0$ implies either $F_{n+1}(\alpha(n + 1)) = 0$ or $F_{n+1}(\beta(n + 1)) = 0$. Similarly $i_3F_{n+1}(\gamma(n + 1)) + i_4F_{n+1}(\xi(n + 1)) = 0$ implies either $F_{n+1}(\gamma(n + 1)) = 0$ or $F_{n+1}(\xi(n + 1)) = 0$.

Suppose that $F_{n+1}(\alpha(n + 1)) = 0$; in the case when either $F_{n+1}(\beta(n + 1)) = 0$, $F_{n+1}(\gamma(n + 1)) = 0$ or $F_{n+1}(\xi(n + 1)) = 0$, the proof is similar. Observe that, by Lemma 4.2, $\alpha(n + 1) = a_nr_1 + b_nr_2$, where $r_1 = w((n + 1)(2c_n + 1))a_n + w((n + 1)(2c_n + 1) - 1)ye_n$. By the

definition of the mapping F_{n+1} we get

$$F_{n+1}(\alpha(n+1)) \in E_{n+1}(a_n)G_{n+1}(w((n+1)(2c_n+1)))E_{n+1}(a_n) + L,$$

where

$$L = E_{n+1}(a_n)G_{n+1}(w((n+1)(2c_n+1)-1)y)E_{n+1}(e_n) + E_{n+1}(b_n)A.$$

By Lemma 5.1, elements $E_{n+1}(a_n)$, $E_{n+1}(b_n)$, and $E_{n+1}(e_n)$ are linearly independent over K and of the same degree. Therefore $G_{n+1}(w((n+1)(2c_n+1))) = 0$, which is a contradiction with Lemma 5.2. \square

LEMMA 5.4. *Let n be a natural number. Then $F_n(w(c_n)) \neq 0$, for every n .*

Proof. It follows by Lemma 5.3, because 0 is linearly dependent over K . \square

Proof of Theorem 1.3. Let $R = A/I$, and let R' be the subalgebra of R generated by x and y . By the definition of the ideal I , all homogeneous elements in R' are nilpotent. By Lemma 3.2, if $r \in I$ and $r \in H(c_n)$ for some n , then $F_n(r) = 0$. By Lemma 5.4, $F_n(w(c_n)) \neq 0$, for all natural n . Therefore $w(c_n) \notin I$ for all natural n . By Lemma 4.5, the element $x + y^2$ is not quasi-regular in A/I . Therefore R' is not Jacobson radical. \square

6. Nilpotent matrices

In Sections 6 and 7 we will use the following notation.

(1) Given a ring R and subsets $A, B \subset R$, denote $AB = \{\sum_i a_i b_i : a_i \in A, b_i \in B\}$, $A + B = \{a + b : a \in A, b \in B\}$. Note that $A + B, AB \subset R$.

(2) Let R be a ring and let S be a subset of R . Moreover, let $P = S + S^2 + \dots$ be the subring of R generated by S .

(3) Let $r \in P$ and $r = \sum_{i=1}^t r_i$ with $r_i \in S^i$. By the degree of r we will denote the smallest number $d = d(r)$ such that $r \in S + S^2 + \dots + S^d$.

(4) Let $S[x]$ be the polynomial ring over S and let $r(x) = \sum_{i=1}^t r_i x^i$. We can write formally $(1 - r(x))^{-1} = 1 + \sum_{i=1}^{\infty} r(x)^i$.

(5) By arranging elements with respect to the degree of x , we obtain

$$(1 - r(x))^{-1} = 1 + \sum_{i=1}^{\infty} v_i(r)x^i,$$

for some $v_i(r) \in P$.

(6) Denote $v_0(r) = 1$ and $v_i(r) = 0$ for $i < 0$.

LEMMA 6.1. *Let $R, S, v_i(r)$ be as in the beginning of this section. Let $r = \sum_{i=1}^t r_i$ with $r_i \in S^i$. Then for every natural number n , $v_n(r) = \sum_{i=1}^t r_i v_{n-i}(r)$ and $v_n(r) = \sum_{i=1}^t v_{n-i}(r)r_i$. Moreover, if $v_n(r) = 0$ for almost all n , then r is quasi-regular, that is, $(1+r)(1+s) = 0$ for some $s \in P$.*

Proof. By the definition of elements $v_i(r)$, we have $1 = (1 - r(x))(1 - r(x))^{-1} = (1 - r(x))(1 + \sum_{i=1}^{\infty} v_i(r)x^i)$. Recall that, $v_0(r) = 1$, and so $r(x)(\sum_{i=0}^{\infty} v_i(r)x^i) = \sum_{i=1}^{\infty} v_i(r)x^i$. By comparing the elements of the degree n on both sides in this equation, we get $v_n(r) = \sum_{i=1}^t r_i v_{n-i}(r)$. Similarly, by comparing the elements of the degree n in the equation $1 = (\sum_{i=0}^{\infty} v_i(r)x^i)(1 - r(x))$, we get that $v_n(r) = \sum_{i=1}^t v_{n-i}(r)r_i$.

Assume now that there is a number m , such that $v_n(r) = 0$ for all $n > m$. Now, observe that $(1 - r(x))^{-1} = 1 + \sum_{i=1}^{\infty} v_i(r)x^i = 1 + \sum_{i=1}^m v_i(r)x^i$. Therefore $(1 - r(x))(1 + \sum_{i=1}^m v_i(r)x^i) = 1$. Let $x = 1$; then $(1 - r)((1 + \sum_{i=1}^m v_i(r))) = 1$. Similarly $((1 + \sum_{i=1}^m v_i(r))(1 - r) = 1$. Hence r is quasi-regular (with $s = -\sum_{i=1}^m v_i(r) \in P$). \square

7. Proof of Theorem 1.2.

In this section we prove Theorem 1.2. Let $r \in P$. We can write $r = \sum_{i=1}^{\gamma} s_i$, for some γ , where each s_i is a product of elements from S . Write $s_i = s_{i,1} \dots s_{i,d_i}$, where d_i is the degree of element s_i and all $s_{i,j} \in S$. Denote $Q = \{(i, j) : 1 \leq i \leq \gamma, 1 \leq j \leq d_i\}$. If $s_i = 0$ we put $d_i = 1$ and $s_{i,1} = 0$.

DEFINITION 7.1. Let $r = \sum_{i=1}^{\gamma} s_i$, S , Q , be as in the beginning of this section. Let X be a matrix with rows and columns enumerated by the set Q . Let $x[(i, j), (i', j')]$ denote the (i, j) , (i', j') entry of the matrix X , that is, the entry in the row indexed by (i, j) and column indexed by (i', j') . Set

- (1) $x[(i, d_i), (i', 1)] = s_{i',1}$, for all $i, i' \leq \gamma$ (recall that d_i is the degree of the element s_i).
- (2) $x[(i, j), (i, j+1)] = s_{i,j+1}$ for all $i \leq \gamma$ and all $j < d_i$.
- (3) All other entries are zero.

REMARK. Denote $S = \{s_{i,j} : (i, j) \in Q\}$. Let v be an infinite word, which is a product of all possible permutations of the elements s_i . Note that the (i, j) , (i', j') entry of the matrix X is equal to $s_{i',j'}$ if $s_{i,j}s_{i',j'}$ is a subword of v , and is zero when $s_{i,j}s_{i',j'}$ is not a subword of v .

Similarly, the (i, j) , (i', j') entry of the matrix X^n will be equal to all products $c_1 \dots c_n$ of n elements $c_i \in S$ such that $c_n = s_{i',j'}$ and $s_{i,j}c_1 \dots c_n$ is a subword of v . This implies Lemma 7.1.

Let $x^n[(i, j), (i', j')]$ denote the (i, j) , (i', j') entry of the matrix X^n , that is, the entry in the row indexed by (i, j) and column indexed by (i', j') in the matrix X^n . Let $v_n(r)$ be defined as in Section 6.

LEMMA 7.1. Let $r = \sum_{i=1}^{\gamma} s_i \in P$, $v_n(r)$, X be as defined above. Let k be a natural number and let $(i, j) \in Q$. Then $x^n[(1, d_1), (i, j)] = v_{n-j}(r)s_{i,1} \dots s_{i,j}$.

Proof. We will proceed by induction on n . If $n = 1$ then the result is true. Assume that the result is true for some n ; we will show it is true for $n + 1$. An elementary fact from the matrix theory shows that, given two indices m, n , the m, n th entry of the matrix AB equals the dot product of the m th row of the matrix A multiplied by the n th column of the matrix B . Set $A = X^n$ and $B = X$, $m = (1, d_1)$, $n = (i, j)$. Now, the entry $x^{n+1}[(1, d_1), (i, j)]$ of the matrix $X^{n+1} = AB$ is the dot product of the $(1, d_1)$ th row $r((1, d_1), X^n)$ of the matrix X^n by the (i, j) th column $c((i, j), X)$ of the matrix X . By the inductive assumption, the row $x^n(1, d_1)$ of the matrix X^n has entries $x^n[(1, d_1), (i, j)] = v_{n-j}(r)s_{i,1} \dots s_{i,j}$.

Suppose first that $(i, j) \in Q$ and $j > 1$. Then (i, j) th column $c((i, j), X)$ of X has only one non-zero entry $x[(i, j-1), (i, j)] = s_{i,j}$. Observe now that $x^{n+1}[(1, d_1), (i, j)] = r((1, d_1), X^n)c((i, j), X)$. Therefore

$$x^{n+1}[(1, d_1), (i, j)] = x^n[(1, d_1), (i, j-1)]x[(i, j-1), (i, j)] = v_{n+1-j}(r)s_{i,1} \dots s_{i,j},$$

as required. Suppose now that $(i, j) \in Q$ and $j = 1$. Consider the entries of the $(i, 1)$ th column $c((i, 1), X)$ of X . Observe that the entries $x^n[(k, d_k), (i, 1)] = s_{i,1}$ for $k \leq \gamma$, and all other entries

in this column are zero. It follows that $x^{n+1}[(1, d_1), (i, 1)] = r((1, d_1), X^n)c((i, 1), X)$, and so $x^{n+1}[(1, d_1), (i, 1)] = \sum_{k=1}^{\gamma} x^n[(1, d_1)(k, d_k)]x[(k, d(k)), (i, 1)]$. By the inductive assumption,

$$x^n[(1, d_1)(k, d_k)] = v_{n-d(k)}(r)s_{k,1} \cdots s_{k,d(k)} = v_{n-d(k)}(r)s_k.$$

Therefore $x^{n+1}[(1, d_1), (i, 1)] = \sum_{k=1}^{\gamma} v_{n-d(k)}(r)s_k s_{i,1}$. Observe that, by Lemma 6.1, $\sum_{k=1}^{\gamma} v_{n-d(k)}(r)s_k = v_n(r)$. Consequently, $x^{n+1}[(1, d_1), (i, 1)] = v_n(r)s_{i,1}$, as required. \square

LEMMA 7.2. *Let $r = \sum_{i=1}^{\gamma} s_i \in P$, $v_i(r)$, X be as above. Assume that $X^n = 0$ for some natural number n . Then $v_i(r) = 0$ for every $i > 2n(\deg r)$. Moreover r is quasi-regular.*

Proof. Let $m > n$. Note that $X^m = 0$ by the assumptions. By Lemma 7.1, $x^m[(1, d_1), (i, d_i)] = v_{m-d_i}(r)s_{i,1} \cdots s_{i,d_i}$, and so $v_{m-d_i}(r)s_i = 0$, for every $i \leq \gamma$. By Lemma 6.1, $\sum_{k=1}^{\gamma} v_{m-d_k}(r)s_k = v_m(r)$. Consequently, $v_m(r) = 0$ for all $m \geq n$. By Lemma 6.1, r is quasi-regular. \square

Proof of Theorem 1.2. By Lemma 7.2, all elements in P are quasi-regular, and so P is Jacobson radical. Note that if $r \in S^j$ for some k , then r is nilpotent because $v_i(r) = r^i$, for every i . Let k be a natural number. Let $R' = M_k(R)$ be the matrix ring over R and let $S' = M_k(S)$. Then R', S' satisfy the assumptions of Theorem 1.2, because R and S satisfy the assumptions of Theorem 1.2. By the remark from the beginning of this proof applied to R', S' instead of R, S , we get that all matrices from $M_n(S')^j$ are nilpotent, which completes the proof. \square

8. Open questions

In this section we mention some open questions related to the Jacobson radical and nil algebras. Every Jacobson radical algebra R forms a group under the operation $a * b = a + b + ab$. Amberg and Kazarin asked whether every nil algebra is nilpotent if its adjoint group has only finitely many generators [1]. Amitsur asked whether every finitely presented (finitely generated modulo a finitely generated ideal) Jacobson radical algebra is nilpotent. A related question was asked by Ufnarovskij ([33, p. 49]).

QUESTION 8.1 (Ufnarovskij). Are finitely presented nil algebras nilpotent?

It was mentioned by Zelmanov that this problem is related to the (open) Burnside problem for finitely presented residually finite groups [35]. Some open questions on nil and nilpotent rings related to group theory and Lie algebras can be found in [1, 12, 35].

It is known that nil ideals in Noetherian rings are nilpotent. There is a related conjecture by Herstein [32].

CONJECTURE 8.1 (Herstein's conjecture). Suppose that $I \subseteq J$ are two left ideals of a left Noetherian ring R and satisfy that J is nil over I ; then J is nilpotent over I .

Stafford proved that the conjecture is true for simple rings and some other classes of rings [32], but the general case remains open. Also, the Jacobson conjecture is still open [18].

CONJECTURE 8.2 (Jacobson's conjecture). If R is a left and right Noetherian ring then $\bigcap_{n \geq 1} (J(R))^n = 0$.

This conjecture is true for some important classes of rings, as shown by Jategonker, Lenagan and others [11, 20]. Not much is known about finitely generated (ungraded) Jacobson radical algebras which are also domains. I don't know any interesting examples of finitely generated domains with non-zero Jacobson radicals. One way of constructing examples of finitely generated Jacobson radical rings is to apply techniques invented by Markov in 1981 [21]. However, rings constructed in this way contain a lot of zero divisors. Bell and Small proved that there are finitely generated primitive algebraic algebras which are infinite-dimensional over their centers [5]. There is a related open question by Small (Private Communication, San Diego, 17 March 2008).

QUESTION 8.2 (Small's question). Are finitely generated simple algebraic algebras finitely dimensional?

Every nil algebra is algebraic. Examples of simple nil algebras were constructed in 2002 [30]. However, by Nakayama's lemma simple nil algebras cannot be finitely generated [17, 18]. It is not known if there are simple nil algebras over uncountable fields. Simple Jacobson radical algebras over arbitrary fields were constructed by Sasiada in 1961 [10]. It was shown in [31] that primitive ideals in graded rings with all homogeneous elements nilpotent are homogeneous. Bergman showed that Jacobson radicals of graded algebras are homogeneous [7]. It is not known if the nil radical (that is, the largest nil ideal) in a graded algebra needs to be homogeneous [24]. A result of Lawrence says that if A and B are K -algebras and neither A nor B has a non-zero algebraic ideal, and K is algebraically closed, then $A \otimes_K B$ is semiprimitive, that is, has zero Jacobson radical [19]. The case when A and B are nil, and hence algebraic, is much more complicated, and not much is known. Some open questions about tensor products of nil algebras can be found in [24]. There are also various open questions about the dimensions of the Jacobson radical algebras; for example, the following is open.

QUESTION 8.3. Are Jacobson radicals of finitely generated algebras with Gelfand–Kirillov dimension two over uncountable fields nilpotent?

Note that a result of Small and Warfield and Begman's gap theorem assures that the Jacobson radical of a finitely generated algebra with Gelfand–Kirillov dimension smaller than two is nilpotent [15].

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